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1997 J. Phys. A: Math. Gen. 30 6925

(http://iopscience.iop.org/0305-4470/30/19/027)

J. Phys. A: Math. Gen. **30** (1997) 6925–6938. Printed in the UK PII: S0305-4470(97)82978-2

KP symmetry reductions and a generalized constraint

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Received 1 April 1997

Abstract. We study the link between symmetry reductions and constraints of the Kadomtsev–Petviashvili equations in terms of the tau function. We propose a generalization—adapted to non-zero boundary conditions—of the standard constraints, and show a particular class of solutions (solitons).

1. Introduction

In recent years, considerable interest has been shown in non-standard reductions of the Kadomtsev–Petviashvili (KP) hierarchy, so-called k-constraints [1–3]. These reductions are a generalization of the standard KP reductions (of which the KdV equation is the best-known example) explaining for instance how the AKNS system may be viewed as a reduction of the KP hierarchy (it is a 1 constraint).

It is not difficult to show a connection between this type of reduction and a so-called symmetry constraint [3, 4]. This link has, however, not been fully explained: it can be proven that the k constraint always implies a symmetry constraint (constraining the kth flow to be equal to a 'squared eigenfunction' symmetry); the converse, however, does only hold in the case of zero boundary conditions for all the fields involved [4]. Since it is well known that, for example, the AKNS system has so-called dark soliton solutions ('dark' meaning non-zero boundary conditions), it is worth investigating the precise link between these two types of reductions in the case of non-vanishing boundary conditions and if possible, to generalize the constraints to this situation.

To accomplish this task we shall first discuss a number of properties for a bilinear potential $\Omega(q, r)$ introduced in [5], associated with KP eigenfunctions q and adjoint eigenfunctions r. Very recently, some of these properties—notably the connection between Ω and vertex operators in Sato theory—have been discussed by other authors [6]. In the present paper we show that this potential plays the role of a symmetry on the level of the KP tau function τ [7] (i.e. the product $\tau\Omega$ is a symmetry for the bilinear KP equations). The KP bilinear identity [7–9] will be shown to give rise to a compact proof of this property for the entire hierarchy. It will also be seen that this potential is *always* the ratio of two KP tau functions. In these proofs a number of results obtained independently from [6] will be presented and proven explicitly. These proofs bring crucial elements to proofs of subsequent theorems presented in this paper. For example, a practical recipe and an explicit expression for the potential in the case of the KP wavefunctions $\psi(\underline{t}, \lambda)$ and $\psi^*(\underline{t}, \lambda)$ (in terms of the KP tau function and the spectral parameter) are given. These results will prove useful later on.

After giving the definition of the (m-vector) k constrained KP hierarchy, we shall use the bilinear formulation of this reduction to show its equivalence with symmetry constraints on the tau function: $\tau_{t_k} = \tau\Omega$. We shall see that this connection between the k constraint and the symmetry constraint allows for a geometric interpretation of the constraints. As the physical variable u is but the second logarithmic derivative of the tau function τ, we shall see that this connection between the symmetry and constraints leaves room for a generalized k constraint. Such a generalization is introduced in the last section of this paper. Since it is very similar to the original k constraint, a bilinear formulation can be found by analogy to the standard case, the solutions we shall give are, however, of an entirely different nature than the bidirectional Wronskians obtained for the standard case in [10]. We limit ourselves to discussing the nature of the soliton solutions and find that they are '$p^k + c/p = q^k + c/q$'-reductions of the KP N-soliton solutions.

2. The KP hierarchy and symmetries

In this section we shall briefly review the Sato framework underlying the KP hierarchy as well as the tau function approach [7–9]. In this theory, the KP hierachy of nonlinear partial differential equations, is described with the help of the (pseudodifferential) gauge operator $P = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \cdots$ (the coefficients w_j depend on the variable $\underline{t} = (t_1 = x, t_2, t_3, \ldots)$) which is required to satisfy Sato's equation (a t_n derivative is denoted by a subscripted t_n):

$$P_{t_n} = -(L^n)_- P. \tag{1}$$

The Lax pseudodifferential operator L is defined by $L \equiv P\partial P^{-1} \equiv \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots$. The functions ψ and ψ^* defined by $\psi(\underline{t}, \lambda) \equiv P \exp\xi(\underline{t}, \lambda)$ and $\psi^*(\underline{t}, \lambda) \equiv P^{*-1} \exp\xi(\underline{t}, \lambda)$ with $\xi(\underline{t}, \lambda) = \sum_{n\geqslant 1} \lambda^n t_n$ then satisfy:

$$L\psi = \lambda\psi \qquad L^*\psi^* = \lambda\psi^* \tag{2}$$

$$\psi_{t_n} = B_n\psi \qquad \psi^*_{t_n} = -B_n^*\psi^*. \tag{3}$$

We shall call them wavefunctions and adjoint wavefunctions. The time evolutions of the wavefunctions are governed by the differential operators B_n defined as the differential part of L^n (for example $B_1 = (L)_+ = \partial_x$, $B_2 = (L^2)_+ = \partial_x^2 + 2u_2, \ldots$). A * denotes the formal adjoint ($\partial^* = -\partial$ and $(AB)^* = B^*A^*$).

The linear equations (2, 3) are compatible under the conditions:

$$L_{t_n} = [B_n, L] \qquad \text{and} \qquad \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m] \qquad \forall n, m. \tag{4}$$

The latter equation (4) (for $n = 2$, $m = 3$) leads to the KP equation in the field u_2 (as the entire hierarchy can be expressed in this variable we shall simply denote it by u):

$$(4u_{t_3} - 12uu_x - u_{3x})_x - 3u_{2t_2} = 0. \tag{5}$$

The set of equations found from the compatibility conditions (4) is called the KP hierarchy after its basic member (5).

From Sato's equation (1) and the definition of the KP wavefunctions, it can be shown that the KP wavefunctions satisfy the following bilinear identity [8]:

$$\mathrm{Res}_{\lambda=\infty}[\psi(\underline{t}, \lambda)\psi^*(\underline{t}', \lambda)] \qquad \forall \underline{t}, \underline{t}'. \tag{6}$$

Equation (6) provides the key to a connection with the bilinear (Hirota) description of the KP hierarchy. It can be shown there exists a so-called tau function $\tau(\underline{t})$ such that the KP wavefunctions have the following representation:

$$\psi(\underline{t},\lambda) = \frac{\tau(\underline{t}-\underline{\epsilon}(\lambda))}{\tau(\underline{t})}\exp\xi(\underline{t},\lambda) \qquad \psi^*(\underline{t},\lambda) = \frac{\tau(\underline{t}+\underline{\epsilon}(\lambda))}{\tau(\underline{t})}\exp-\xi(\underline{t},\lambda). \tag{7}$$

Here we have used the notation $\underline{\epsilon}(\lambda) = (\lambda^{-1}, \lambda^{-2}/2, \lambda^{-3}/3, \ldots)$.

The bilinear identity (6) then becomes:

$$\mathrm{Res}_\lambda[\tau(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}+\underline{\epsilon}(\lambda))\exp\xi(\underline{t}-\underline{t}',\lambda)] = 0. \tag{8}$$

This single equation for the KP tau function τ is equivalent to an infinite number of partial differential equations generated by:

$$\mathrm{e}^{\sum_{i=1}^{\infty} y_i D_i}\sum_{j=0}^{\infty} p_j(-2\underline{y})p_{j+1}(\tilde{D})\tau\tau = 0 \qquad \forall\underline{y}. \tag{9}$$

The Schur polynomials $p_i(\underline{t})$ are defined by $\sum_{i=0}^{\infty} p_i(\underline{t})\lambda^i = \exp\xi(\underline{t},\lambda)$. D_i is the well-known Hirota operator with respect to t_i [11].

The simplest equation among the equations (9) is the Hirota form of the KP equation (5):

$$(4D_1D_3 - 3D_2^2 - D_1^4)\tau\cdot\tau = 0 \tag{10}$$

where u and τ are related to each other by $u = \partial_x^2\log\tau$.

The linear equations for the time-evolutions can also be written under the form of a residue; let q satisfy the linear equations $q_{t_n} = B_n q$ (although it is *not* required to satisfy an eigenvalue problem as, for example, (2), we shall call this an 'eigenfunction') then the following relation holds [7]:

$$\mathrm{Res}_\lambda[\lambda\psi(\underline{t},\lambda)\psi^*(\underline{t}',\lambda)q(\underline{t}-\underline{\epsilon}(\lambda))] = 0. \tag{11}$$

Introducing ρ by $q = \rho/\tau$, one can rewrite this equation:

$$\mathrm{Res}_\lambda[\lambda\rho(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}+\underline{\epsilon}(\lambda))\exp\xi(\underline{t}-\underline{t}',\lambda)] = 0 \tag{12}$$

which are known as the modified KP equations. There exists an alternative (but equivalent) formulation of the linear equations $q_{t_n} = B_n q$, namely [10]:

$$\mathrm{Res}_\lambda[\lambda^{-1}\psi(\underline{t},\lambda)\psi^*(\underline{t}',\lambda)q(\underline{t}'+\underline{\epsilon}(\lambda))] = q(\underline{t}). \tag{13}$$

Analogously, a function r ('adjoint eigenfunction') satisfying the adjoint time-evolution equations $-r_{t_n} = B_n^* r$ satisfies the following equations:

$$\mathrm{Res}_\lambda[\lambda\psi(\underline{t},\lambda)\psi^*(\underline{t}',\lambda)r(\underline{t}'+\underline{\epsilon}(\lambda))] = 0 \tag{14}$$

$$\mathrm{Res}_\lambda[\lambda^{-1}\psi(\underline{t},\lambda)\psi^*(\underline{t}',\lambda)r(\underline{t}-\underline{\epsilon}(\lambda))] = r(\underline{t}'). \tag{15}$$

Given a solution q to the linear equations $q_{t_n} = B_n q$ and a solution r to the adjoint linear equations $r_{t_n} = -B_n^* r$, it is not difficult to see that the t_n derivative of the product qr is always a total x derivative ($B_n = \sum_{j=1}^{n} b_{n,j}\partial_x^j$):

$$(qr)_{t_n} = \left(\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{i+1}(rb_{n,j})_{(i-1)x}q_{(j-i)x}\right)_x \equiv (A_n(q,r))_x. \tag{16}$$

Hence, the idea of defining a 'potential' $\Omega(q,r)$ by the differential [5]:

$$\mathrm{d}\Omega \equiv \sum_{n\geqslant 1} A_n(q,r)\mathrm{d}t_n = qr\,\mathrm{d}x + (q_x r - qr_x)\,\mathrm{d}t_2 + \cdots. \tag{17}$$

(From (4), it is possible to check that $\mathrm{d}\Omega$ is a total differential, i.e. $(\Omega_{t_n})_{t_m} = (\Omega_{t_m})_{t_n}$.) The potential $\Omega(q, r)$ is only defined modulo an arbitrary integration constant (independent of $x, t_2, t_3, \ldots$).

For example, let us compute $\Omega(\psi(\underline{t}, \lambda), \psi^*(\underline{t}, \lambda))$ in the special case of the constant tau function ($u = 0$). Then one has $B_n = \partial_x^n$ and $\Omega = \sum_{n\geqslant 1} n t_n \lambda^{n-1} + C$. We shall generalize this result to non-constant tau functions later.

Since we shall be working in a tau function formalism, it is necessary to have some knowledge concerning the action of a shift operator ($\underline{t} \to \underline{t} \pm \underline{\epsilon}(k)$) on such a potential. The key result is:

Property 1. Denoting $\Omega(q, r)$ by $\Omega(\underline{t})$:

$$\Omega(\underline{t} - \underline{\epsilon}(k)) = \Omega(\underline{t}) - k^{-1} q(\underline{t}) r(\underline{t} - \underline{\epsilon}(k)) \tag{18}$$

$$\Omega(\underline{t} + \underline{\epsilon}(k)) = \Omega(\underline{t}) + k^{-1} q(\underline{t} + \underline{\epsilon}(k)) r(\underline{t}). \tag{19}$$

Proof. Since (18) and (19) are the same identity, we shall only prove (18). Taking $\underline{t}' = \underline{t} - \underline{\epsilon}(k)$ one finds that equation (12) implies:

$$\tau(\underline{t})\rho(\underline{t} - \underline{\epsilon}(k)) + k^{-1}\rho_x(\underline{t})\tau(\underline{t} - \underline{\epsilon}(k)) - \rho(\underline{t})\tau(\underline{t} - \underline{\epsilon}(k)) - k^{-1}\rho(\underline{t})\tau_x(\underline{t} - \underline{\epsilon}(k)) = 0 \tag{20}$$

or equivalently:

$$q(\underline{t} - \underline{\epsilon}(k)) = q(\underline{t}) - k^{-1} q_x(\underline{t}) + k^{-1} q(\underline{t}) \left(\frac{\tau_x(\underline{t} - \underline{\epsilon}(k))}{\tau(\underline{t} - \underline{\epsilon}(k))} - \frac{\tau_x(\underline{t})}{\tau(\underline{t})} \right). \tag{21}$$

Similarly, one finds for $r(\underline{t})$ the relation:

$$r(\underline{t} - \underline{\epsilon}(k)) = r(\underline{t}) - k^{-1} r_x(\underline{t} - \underline{\epsilon}(k)) + k^{-1} r(\underline{t} - \underline{\epsilon}(k)) \left(\frac{\tau_x(\underline{t})}{\tau(\underline{t})} - \frac{\tau_x(\underline{t} - \underline{\epsilon}(k))}{\tau(\underline{t} - \underline{\epsilon}(k))} \right). \tag{22}$$

Making use of equation (21), $\Omega(\underline{t} - \underline{\epsilon}(k))$ becomes:

$$\begin{aligned} \Omega(\underline{t} - \underline{\epsilon}(k)) &= \int^x q(\underline{t} - \underline{\epsilon}(k)) r(\underline{t} - \underline{\epsilon}(k)) \\ &= \int^x \left[q(\underline{t}) - k^{-1} q_x(\underline{t}) + k^{-1} q(\underline{t}) \left(\frac{\tau_x(\underline{t} - \underline{\epsilon}(k))}{\tau(\underline{t} - \underline{\epsilon}(k))} - \frac{\tau_x(\underline{t})}{\tau(\underline{t})} \right) \right] r(\underline{t} - \underline{\epsilon}(k)). \end{aligned} \tag{23}$$

Using relation (22) to simplify the last term in the previous expression, one has:

$$\begin{aligned} \Omega(\underline{t} - \underline{\epsilon}(k)) &= \int^x [q(\underline{t}) - k^{-1} q_x(\underline{t})] r(\underline{t} - \underline{\epsilon}(k)) \\ &\quad - q(\underline{t})[r(\underline{t} - \underline{\epsilon}(k)) - r(\underline{t}) + k^{-1} r_x(\underline{t} - \underline{\epsilon}(k))] \\ &= \int^x q(\underline{t}) r(\underline{t}) - k^{-1}[q_x(\underline{t}) r(\underline{t} - \underline{\epsilon}(k)) + q(\underline{t}) r_x(\underline{t} - \underline{\epsilon}(k))] \\ &= \Omega(t) - k^{-1} q(\underline{t}) r(\underline{t} - \underline{\epsilon}(k)) \end{aligned} \tag{24}$$

which proves relation (18). □

Since the only function invariant under a shift $\underline{t} \to \underline{t} \pm \underline{\epsilon}(k)$ is constant, these formulae (18), (19) provide us with a characterization of Ω and a practical recipe for finding $\Omega(q, r)$ from q and r:

$$\Omega(q, r) = f(\underline{t}) \Leftrightarrow f(\underline{t} - \underline{\epsilon}(k)) = f(\underline{t}) - k^{-1} q(\underline{t}) r(\underline{t} - \underline{\epsilon}(k)). \tag{25}$$

As an example of this characterization, we shall show the following expressions:

Property 2.

$$\Omega(\psi(\underline{t},\lambda),r) = \lambda^{-1}\psi(\underline{t},\lambda)r(\underline{t}-\underline{\epsilon}(\lambda)) + C \tag{26}$$

$$\Omega(q,\psi^*(\underline{t},\lambda)) = -\lambda^{-1}\psi^*(\underline{t},\lambda)q(\underline{t}+\underline{\epsilon}(\lambda)) + C. \tag{27}$$

To make use of characterization (25) in the case (26), one needs to show (using the representation (7) of $\psi(\underline{t},\lambda)$ in terms of the KP tau function and $r = \sigma/\tau$):

$$\begin{aligned}(k-\lambda)\sigma(\underline{t}-\underline{\epsilon}(\lambda)-\underline{\epsilon}(k))\tau(\underline{t}) - k\sigma(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}-\underline{\epsilon}(k))\\ +\lambda\sigma(\underline{t}-\underline{\epsilon}(k))\tau(\underline{t}-\underline{\epsilon}(\lambda)) = 0.\end{aligned} \tag{28}$$

Taking $\underline{t}' = \underline{t} - \underline{\epsilon}(k_1) - \underline{\epsilon}(k_2)$ in the bilinear modified KP equation (14), one has:

$$\mathrm{Res}_\lambda\left[\lambda\tau(\underline{t}-\underline{\epsilon}(\lambda))\sigma(\underline{t}-\underline{\epsilon}(k_1)-\underline{\epsilon}(k_2)+\underline{\epsilon}(\lambda))\left(\frac{k_2}{1-\lambda/k_1} - \frac{k_1}{1-\lambda/k_2}\right)\right] = 0 \tag{29}$$

or explicitly:

$$\begin{aligned}k_1\tau(\underline{t}-\underline{\epsilon}(k_1))\sigma(\underline{t}-\underline{\epsilon}(k_2)) - k_1\tau(\underline{t})\sigma(\underline{t}-\underline{\epsilon}(k_1)-\underline{\epsilon}(k_2))\\ -k_2\tau(\underline{t}-\underline{\epsilon}(k_2))\sigma(\underline{t}-\underline{\epsilon}(k_2)) + k_2\tau(\underline{t})\sigma(\underline{t}-\underline{\epsilon}(k_1)-\underline{\epsilon}(k_2)) = 0.\end{aligned} \tag{30}$$

But this is just equation (28) with $\lambda = k_1$ and $k = k_2$. The formula (27) is verified just as easily. □

Formulae (26) and (27) can be explicitly written as ($q = \rho/\tau, r = \sigma/\tau$):

$$\Omega(\psi(\underline{t},\lambda),r) = \lambda^{-1}\sigma(\underline{t}-\underline{\epsilon}(\lambda))/\tau(\underline{t})\exp\xi(\underline{t},\lambda) + C \tag{31}$$

$$\Omega\left(q,\psi^*(\underline{t},\lambda)\right) = -\lambda^{-1}\rho(\underline{t}+\underline{\epsilon}(\lambda))/\tau(\underline{t})\exp-\xi(\underline{t},\lambda) + C. \tag{32}$$

Plugging the formulae (26) and (27) (at $C = 0$ with this potential written as Ω_0) in the alternative modified KP equations (13) and (15), one finds:

$$\mathrm{Res}_\lambda[\psi(\underline{t},\lambda)\Omega_0(q(\underline{t}'),\psi^*(\underline{t}',\lambda))] = -q(\underline{t}) \tag{33}$$

$$\mathrm{Res}_\lambda[\psi^*(\underline{t}',\lambda)\Omega_0(\psi(\underline{t},\lambda),r(\underline{t}))] = r(\underline{t}'). \tag{34}$$

An important instance of such a potential is given by the choice of $\psi(\underline{t},\lambda)$ as q and $\psi^*(\underline{t},\lambda)$ as r; in this special case the associated potential can be expressed in terms of the tau function τ and the spectral parameter λ as:

Property 3.

$$\Omega(\psi(\underline{t},\lambda);\psi^*(\underline{t},\lambda)) = \sum_{n=1}^{\infty}\frac{\tau_{t_n}}{\tau}\lambda^{-n-1} + \sum_{n=1}^{\infty}nt_n\lambda^{n-1} + C \tag{35}$$

where C is an arbitrary constant.

Proof. Let us check that the right-hand side of (35) satisfies equation (18). Using the representation of ψ and ψ^* in terms of the tau function, one finds this reduces to:

$$\begin{aligned}(k-\lambda)\sum_{n=1}^{\infty}(\tau(\underline{t}-\underline{\epsilon}(k))\tau_{t_n}(\underline{t}) - \tau(\underline{t})\tau_{t_n}(\underline{t}-\underline{\epsilon}(k)))\lambda^{-n-1}\\ = \tau(\underline{t}-\underline{\epsilon}(\lambda))\tau(\underline{t}+\underline{\epsilon}(\lambda)-\underline{\epsilon}(k)) - \tau(\underline{t})\tau(\underline{t}-\underline{\epsilon}(k)).\end{aligned} \tag{36}$$

It is easy to check that:

$$\sum_{n=1}^{\infty}\lambda^{-n-1}\tau_{t_n}(\underline{t}) = \lim_{\mu\to\lambda}\frac{\tau(\underline{t}-\underline{\epsilon}(\lambda)+\underline{\epsilon}(\mu)) - \tau(\underline{t})}{\lambda-\mu} \tag{37}$$

and hence also:

$$\sum_{n=1}^{\infty} \lambda^{-n-1} \tau_{t_n}(\underline{t} - \underline{\epsilon}(k)) = \lim_{\mu\to\lambda} \frac{\tau(\underline{t} - \underline{\epsilon}(k) - \underline{\epsilon}(\lambda) + \underline{\epsilon}(\mu)) - \tau(\underline{t} - \underline{\epsilon}(k))}{\lambda - \mu}. \tag{38}$$

Using these two equalities, the left-hand side of equation (36) becomes:

$$\lim_{\mu\to\lambda} \frac{k-\lambda}{\lambda-\mu} (\tau(\underline{t} - \underline{\epsilon}(\lambda) + \underline{\epsilon}(\mu))\tau(\underline{t} - \underline{\epsilon}(k)) - \tau(\underline{t} - \underline{\epsilon}(k) - \underline{\epsilon}(\lambda) + \underline{\epsilon}(\mu))\tau(\underline{t})). \tag{39}$$

Taking $\underline{t} = \underline{x} + \underline{\epsilon}(k_1)$ and $\underline{t}' = \underline{x} - \underline{\epsilon}(k_2) - \underline{\epsilon}(k_3)$ in the KP bilinear identity (8), one finds:

$$\begin{aligned}(k_2 - k_3)\tau(\underline{x})\tau(\underline{x} + \underline{\epsilon}(k_1) - \underline{\epsilon}(k_2) - \underline{\epsilon}(k_3)) - (k_1 - k_3)\tau(\underline{x} + \underline{\epsilon}(k_1) - \underline{\epsilon}(k_2))\tau(\underline{x} - \underline{\epsilon}(k_3))\\ +(k_1 - k_2)\tau(\underline{x} + \underline{\epsilon}(k_1) - \underline{\epsilon}(k_3))\tau(\underline{x} - \underline{\epsilon}(k_2)).\end{aligned} \tag{40}$$

Relabelling $\underline{x} \to \underline{t}$, $k_1 \to \mu$, $k_2 \to \lambda$ and $k_3 \to k$, this can be written as:

$$\begin{aligned}(k-\mu)[\tau(\underline{t} - \underline{\epsilon}(\lambda) + \underline{\epsilon}(\mu))\tau(\underline{t} - \underline{\epsilon}(k)) - \tau(\underline{t} - \underline{\epsilon}(k) - \underline{\epsilon}(\lambda) + \underline{\epsilon}(\mu))\tau(\underline{t})]\\ = (\lambda-\mu)[\tau(\underline{t} + \underline{\epsilon}(\mu) - \underline{\epsilon}(k))\tau(\underline{t} - \underline{\epsilon}(\lambda)) - \tau(\underline{t})\tau(\underline{t} + \underline{\epsilon}(\mu) - \underline{\epsilon}(\lambda) - \underline{\epsilon}(k))]\end{aligned} \tag{41}$$

and hence the limit (39) becomes:

$$\begin{aligned}\lim_{\mu\to\lambda} \frac{k-\lambda}{k-\mu} (\tau(\underline{t} + \underline{\epsilon}(\mu) - \underline{\epsilon}(k))\tau(\underline{t} - \underline{\epsilon}(\lambda)) - \tau(\underline{t})\tau(\underline{t} + \underline{\epsilon}(\mu) - \underline{\epsilon}(\lambda) - \underline{\epsilon}(k)))\\ = \tau(\underline{t} + \underline{\epsilon}(\lambda) - \underline{\epsilon}(k))\tau(\underline{t} - \underline{\epsilon}(\lambda)) - \tau(\underline{t})\tau(\underline{t} - \underline{\epsilon}(k)).\end{aligned} \tag{42}$$

This proves that the representation (35) satisfies the characterization (18). □

Representation (35) could also have been obtained from (26), by choosing $r = \psi^*(\underline{t}, \mu)$, $C = 1/(\lambda - \mu)$ and taking the limit $\mu \to \lambda$.

It is well known (and for that matter easy to check) that the 'squared eigenfunction' $(qr)_x$ is a symmetry for the KP equation (5): $u + \eta(qr)_x$ satisfies the KP equation (5) up to first order in η. Since $u = \partial_x^2 \log\tau$, on the level of the tau function this corresponds to:

$$u + \eta(qr)_x \leftrightarrow \tau \exp \eta \int^x qr = \tau + \eta\tau \int^x qr + \mathcal{O}(\eta^2). \tag{43}$$

Whence, the following theorem.

Theorem 1. The product $\tau\,\Omega(q, r)$ is a symmetry for all KP equations, i.e. $\tau + \eta\tau\Omega(q, r)$ satisfies the KP bilinear identity up to first order in η:

$$\mathrm{Res}_\lambda[\tau(\underline{t} - \underline{\epsilon}(\lambda))\tau(\underline{t} + \underline{\epsilon}(\lambda))(\Omega(\underline{t} - \underline{\epsilon}(\lambda)) + \Omega(\underline{t}' + \underline{\epsilon}(\lambda)))\mathrm{e}^{\xi(\underline{t}-\underline{t}',\lambda)}] = 0. \tag{44}$$

It is easy to prove this property by substituting the relations (18), (19) and using the KP bilinear identity together with the alternative representation of the modified KP equations (13) and (15). □

It is also immediately clear from equation (44), that $\tau(\Omega(q, r) + C + \sum_n c_n t_n)$ is another symmetry of KP. The constant C corresponds to the invariance of the bilinear KP identity under a rescaling of the tau function; the symmetry $\tau \sum_n c_n t_n$ corresponds to a multiplication (to the right) of the gauge operator P by a constant coefficient operator (which leaves Sato's equation (1) invariant). This is another way of saying that the bilinear KP equations (9) are invariant under the transformation $\tau \to \tau \exp \sum_{n\geqslant 1} c_n t_n$.

One can show that an even stronger property holds for the product $\hat\tau \equiv \tau\Omega(q, r)$;

Theorem 2. $\hat\tau$ is a KP tau function:

$$\mathrm{Res}_\lambda[\hat\tau(\underline{t} - \underline{\epsilon}(\lambda))\hat\tau(\underline{t} + \underline{\epsilon}(\lambda))\mathrm{e}^{\xi(\underline{t}-\underline{t}',\lambda)}] = 0. \tag{45}$$

Proof. Substituting $\hat{\tau} = \tau\,\Omega$, using properties (18) and (19), the KP bilinear identity (8) and the (alternative) modified KP identities (13), (15), this reduces to:

$$\Omega(\underline{t}) - \Omega(\underline{t}') = \mathrm{Res}_\lambda[\lambda^{-2}\psi(\underline{t},\lambda)\psi^*(\underline{t}',\lambda)r(\underline{t}-\underline{\epsilon}(\lambda))q(\underline{t}'+\underline{\epsilon}(\lambda))]. \tag{46}$$

Using the relations (26) and (27) (at $C = 0$), the right-hand side is seen to equal:

$$-\,\mathrm{Res}_\lambda[\Omega_0(q(\underline{t}'),\psi^*(\underline{t}',\lambda))\Omega_0(\psi(\underline{t},\lambda),r(\underline{t}))] \equiv I(\underline{t},\underline{t}'). \tag{47}$$

To compute this residue $I(\underline{t},\underline{t}')$, let us calculate

$$\partial_{t_n}\partial_{t'_m} I(\underline{t},\underline{t}') = \text{ (differential operator acting on)}\,\mathrm{Res}\left[\psi(\underline{t},\lambda)\psi^*(\underline{t}',\lambda)\right] = 0. \tag{48}$$

Hence we see that $I(\underline{t},\underline{t}') = f(\underline{t}) + g(\underline{t}')$. Explicitly computing $I(\underline{t},\underline{t}' = \underline{t})$ with the help of formulae (31) and (32), one finds $I(\underline{t},\underline{t}' = \underline{t}) = 0$; hence $I(\underline{t},\underline{t}') = f(\underline{t}) - f(\underline{t}')$. Since

$$\begin{aligned}[\partial_{z_n} I(\underline{t}-\underline{z},\underline{t}+\underline{z})]_{\underline{z}=0} = -2\partial_{t_n} f(\underline{t}) &= -\,\mathrm{Res}_\lambda[\Omega_0(q(\underline{t}),\psi^*(\underline{t},\lambda))\Omega_{0,t_n}(\psi(\underline{t},\lambda),r(\underline{t}))]\\ &\quad + \mathrm{Res}_\lambda[\Omega_{0,t_n}(q(\underline{t}),\psi^*(\underline{t},\lambda))\Omega_0(\psi(\underline{t},\lambda),r(\underline{t}))].\end{aligned} \tag{49}$$

Since by definition $\partial_{t_n}\Omega(u,v) = A_n(u,v) \equiv \sum_{i,j=0}^{n-1} a_{ij}u_{ix}v_{jx}$ (see (17)), we have

$$\begin{aligned}-2\partial_{t_n} f(\underline{t}) &= -\sum_{i,j} a_{ij}\,\mathrm{Res}_\lambda[\partial_x^i\psi(\underline{t},\lambda)\Omega_0(q(\underline{t}),\psi^*(\underline{t},\lambda))]\partial_x^j r(\underline{t})\\ &\quad + \sum_{i,j} a_{ij}\partial_x^i q(\underline{t})\,\mathrm{Res}_\lambda[\partial_x^j\psi^*(\underline{t},\lambda)\Omega_0(\psi(\underline{t},\lambda),r(\underline{t}))]\\ &= \sum_{i,j} a_{ij}q_{ix}r_{jx} + \sum_{i,j} a_{ij}q_{ix}r_{jx} = 2A_n(q,r) = 2\partial_{t_n}\Omega(q,r)\end{aligned} \tag{50}$$

where we have used (33) and (34). We may conclude that $f(\underline{t}) = -\Omega(q,r)$. □

This proof is very similar to the proof of theorem 1 in [12].

Theorem 2 states that one can always write $\Omega(q,r)$ as a ratio of two tau functions: $\Omega(q,r) = \hat{\tau}/\tau$. For instance, in the case of the Wronskian-type solution $q = \rho/\tau$, $r = \sigma/\tau$ with $\sigma = W[\varphi_1,\ldots,\varphi_{N-1}]$, $\tau = W[\varphi_1,\ldots,\varphi_{N-1},\varphi_N]$ and $\rho = W[\varphi_1,\ldots,\varphi_{N-1},\varphi_N,\varphi_{N+1}]$ one finds $\hat{\tau} = W[\varphi_1,\ldots,\varphi_{N-1},\varphi_{N+1} + C\varphi_N]$, again a KP-type Wronskian. In the case of the KP wavefunctions ψ and adjoint wavefunctions ψ^*, $\hat{\tau}$ can be written as:

$$\begin{aligned}\lim_{\mu\to\lambda}\frac{\tau(\underline{t}+\underline{\epsilon}(\mu)-\underline{\epsilon}(\lambda))\mathrm{e}^{\xi(\underline{t},\lambda)}\mathrm{e}^{-\xi(\underline{t},\mu)} - \tau(\underline{t})}{\lambda-\mu} &= \lim_{\mu\to\lambda}\frac{[1-X(\lambda,\mu)]\tau(\underline{t})}{\mu-\lambda}\\ &= \lim_{\mu\to\lambda}\frac{\exp[-X(\lambda,\mu)]\tau(\underline{t})}{\mu-\lambda}\end{aligned} \tag{51}$$

(where $X(p,q)$ is the well-known vertex operator) which is a KP tau function as explained in [8]. In general $\Omega(\psi(\underline{t},\lambda),\psi^*(\underline{t},\mu))$ can be written with the help of the vertex operator $X(\lambda,\mu)$ acting on τ.

The first theorem is actually a special case of the second one, since Ω is only defined up to a constant, we have just seen that $\hat{\tau} + C\tau$ is a KP tau function. Hence $\tau + \eta\hat{\tau}$ solves the bilinear KP equation up to all orders of η ($\eta = C^{-1}$)!

3. Constraints and symmetry reductions

The (m-vector) k constrained KP hierarchy [1, 2] is the reduction of KP obtained by imposing the following condition on the Lax operator L:

$$L^k = B_k + \sum_{i=1}^{m} q_i\partial^{-1}r_i \tag{52}$$

for some positive integer $k(1, 2, \ldots)$, where the $2m$ auxiliary functions q_i and r_i satisfy the KP linear equations

$$q_{i,t_n} = B_n q_i \qquad r_{i,t_n} = -B_n^* r_i \qquad \forall n, i : 1 \ldots m. \tag{53}$$

The constraint (52) reduces the $(2+1)$-dimensional KP equations to integrable $(1+1)$-dimensional systems. The simplest example is the case $k = 1$ and $m = 1$, where relation (52) becomes:

$$\partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots = \partial + qr\partial^{-1} - qr_x\partial^{-2} + \cdots \tag{54}$$

from which we have that $u_2 = qr$. Equations (53) (at $n = 2$) become:

$$q_{t_2} = q_{2x} + 2q^2 r \qquad - r_{t_2} = r_{2x} + 2qr^2 \tag{55}$$

which is easily recognized as the AKNS system.

In [10, 13, 12, 14], a general class of solutions was described by using several direct methods one of them using the bilinear formulation of constraint (52) [2]:

$$\mathrm{Res}_\lambda[\lambda^k \psi(\underline{t}, \lambda)\psi^*(\underline{t}', \lambda)] = \sum_{i=1}^{m} q_i(\underline{t}) r_i(\underline{t}'). \tag{56}$$

These solutions are represented using bidirectional Wronskians (for the functions τ, $\rho_i = \tau q_i$ and $\sigma_i = \tau r_i$).

It is well known [3] that the constraint (52) implies:

$$u_{t_k} = \sum_{i=1}^{m} (q_i r_i)_x. \tag{57}$$

This expression is called a symmetry constraint since both the left- and right-hand side of the equality are symmetries of the KP equation (5). The converse statement (symmetry constraints imply k constraints) does not hold as we shall see when considering a generalized constraint; the reason being the fact that there is an underlying more fundamental relation for the tau function of which equation (57) is a mere differential consequence. Let us now derive this relation.

It was established in [1, 2] that the relation (52) is equivalent to (cf also equation (56)):

$$\sum_{i=1}^{m} q_i r_{i,jx} = \mathrm{Res}_\lambda[\lambda^k \psi(\underline{t}, \lambda)\psi^*_{jx}(\underline{t}, \lambda)] \qquad \forall j. \tag{58}$$

It follows from the definition (17) that this is equivalent to

$$\sum_{i=1}^{m} \Omega_{t_n}(q_i, r_i) = \mathrm{Res}_\lambda\left[\lambda^k \Omega_{t_n}(\psi(\underline{t}, \lambda), \psi^*(\underline{t}, \lambda))\right] \qquad \forall n. \tag{59}$$

Using the explicit expression (35) for $\Omega(\psi, \psi^*)$, one finds:

$$\partial_{t_n} \sum_{i=1}^{m} \Omega(q_i, r_i) = \partial_{t_n}\partial_{t_k} \log\tau \qquad \forall n \tag{60}$$

or:

$$\sum_{i=1}^{m} \tau\Omega(q_i, r_i) = \tau_{t_k} \tag{61}$$

where we have absorbed a possible integration constant in one of the potentials $\Omega(q_i, r_i)$.

Considering relation (61) (noting that τ_{t_k} is a symmetry), we can say that the k constrained KP hierarchy is a symmetry reduction of the KP hierarchy (left- and right-hand sides of relation (61) are symmetries of the KP bilinear identity (8)): there is an *equivalence* between the constraint (52) and the symmetry reduction (61).

It is quite clear that relation (57) is but the mere second x derivative of this equation. We shall see later on how this will make it possible to have a different constraint on L (of type (52)) still implying relation (57) but not implying relation (61). In [4], it was shown that there is equivalence between the constraint (52) and the reduction (57) in the case that the eigenfunctions q and r satisfy zero-boundary conditions at infinity. The generalized constraint we shall introduce in section 4 will therefore be adapted to the case of non-vanishing boundary conditions.

Using theorem 2, we also see from equation (61) that imposing a k constraint on the KP hierarchy implies that the t_k derivative of the KP tau function is a linear combination of m KP tau functions.

$$\sum_{i=1}^{m} \hat{\tau}_i = \tau_{t_k}. \tag{62}$$

This is the geometric interpretation of the k constraint (and hence also of the symmetry constraint) [16, 17]. The corresponding property in the case of the standard k reduction [7, 8] is that $\tau_{t_k} = C\tau$ (notice that the standard k reduction is just the case $q_i = r_i = 0$ of the k constraint and that $\Omega(0, 0) = C$).

4. A generalized constraint

In this section we shall introduce a generalized constraint for the KP hierarchy. This constraint has a very similar form when written in terms of a condition on the Lax operator L. The condition we impose will include a constant c which will be related to the asymptotic value (at infinity) of the auxiliary fields q and r. In fact this new constraint is particularly well suited to treat systems with non-zero boundary conditions (but where the field $u = \partial_x^2 \log \tau$ still has zero-boundary conditions; e.g. solitons of the sech-squared type). As a very special example we shall be able to derive the dark soliton solutions of the nonlinear Schrödinger equation. Since we are dealing here with a variation of the classical k constraint, most proofs are simple generalizations (e.g. bilinear form, interpretation as symmetry constraint). The proof of the explicit solutions is also similar to [10]; however, we only obtain soliton solutions.

4.1. Definition

A slight variation on the k constraint (52) is the following c-k constraint:

$$L^k = B_k + q\partial^{-1}r - cL^{-1} \tag{63}$$

where c is some, *a priori*, constant. (It is a well known fact that every (non-zero) pseudodifferential operator has an inverse, e.g. the inverse of L is $L^{-1} = \partial^{-1} - u_2\partial^{-3} + (u_{2,x} - u_3)\partial^{-4} + \cdots$.) When c is chosen zero, one recovers the classical k constraint in the scalar case ($m = 1$).

Let us look at the example $k = 1$. One finds that (63) implies that $qr = u + c$ such that the t_2-time evolutions for q and r become

$$q_{t_2} = q_{2x} + 2(qr - c)q \qquad \text{and} \qquad -r_{t_2} = r_{2x} + 2(qr - c)r \tag{64}$$

which is the nonlinear Schrödinger equation under the condition $q = r^*$ and $t_2 \to \mathrm{i}t_2$. Since the field u typically vanishes at infinity (e.g. sech squared solitons) one finds that the fields q and r typically have non-zero (c) boundary conditions at infinity.

4.2. Bilinear forms and connection with symmetry constraints

Let us first look for a bilinear formulation of this c-k constraint. From the definition $q\partial^{-1}r = qr\partial^{-1} - qr_x\partial^{-2} + qr_{2x}\partial^{-3} + \cdots$ and relation (63), it follows that ($j \geqslant 0$):

$$\mathrm{Res}_\partial[(L^k + cL^{-1})\partial^j] = \mathrm{Res}_\partial[B_k\partial^j + q\partial^{-1}r\partial^j] = (-1)^j q r_{jx} \tag{65}$$

and, using lemma 7.3.2 in [17], one finds:

$$\mathrm{Res}_\lambda[(\lambda^k + c\lambda^{-1})\psi(\underline{t},\lambda)\psi^*_{jx}(\underline{t},\lambda)] = q(\underline{t})r_{jx}(\underline{t}) \tag{66}$$

and hence:

$$\mathrm{Res}_\lambda[(\lambda^k + c\lambda^{-1})\psi(\underline{t},\lambda)\psi^*(\underline{t}',\lambda)] = q(\underline{t})r(\underline{t}') \tag{67}$$

or equivalently ($q = \rho/\tau, r = \sigma/\tau$):

$$\mathrm{Res}_\lambda[(\lambda^k + c\lambda^{-1})\tau(\underline{t} - \underline{\epsilon}(\lambda))\tau(\underline{t}' + \underline{\epsilon}(\lambda))\mathrm{e}^{\xi(\underline{t}-\underline{t}',\lambda)}] = \rho(\underline{t})\sigma(\underline{t}'). \tag{68}$$

These last equations (together with (12) and (14)) form the bilinear representation of the c-k constraint. One immediate use can be found in proving the existence of N-soliton solutions to this generalized constraint (this topic will be addressed in section 4.3).

Let us now look at the links with a possible symmetry constraint. To find such a corresponding symmetry constraint, we only need to adapt the previous case ($c = 0$). Repeating the calculation (58)–(61) for expression (66), one easily finds that equation (63) is equivalent to:

$$\Omega_{t_n}(q,r) = \mathrm{Res}_\lambda[(\lambda^k + c\lambda^{-1})\Omega_{t_n}(\psi(\underline{t},\lambda),\psi^*(\underline{t},\lambda))] \qquad \forall n \tag{69}$$

or

$$\Omega(q,r) = \partial_{t_k}\log\tau + cx. \tag{70}$$

Hence we see that the generalized constraint (63) is equivalent to a symmetry constraint (as seen in section 2, both τ_{t_k} and $\tau(\Omega - cx)$ are symmetries). Let us stress again that the generalized constraint (63) also implies the relation (57).

Since $\hat{\tau} = \Omega\tau$ is a KP τ function, we see that τ satisfies the additional equation:

$$\begin{aligned}\mathrm{Res}_\lambda[(\tau_{t_k}(\underline{t} - \underline{\epsilon}(\lambda)) + c(x - \lambda^{-1})\tau(\underline{t} - \underline{\epsilon}(\lambda)))\\ \times(\tau_{t_k}(\underline{t}' + \underline{\epsilon}(\lambda)) + c(x' + \lambda^{-1})\tau(\underline{t}' + \underline{\epsilon}(\lambda)))\exp\xi(\underline{t} - \underline{t}',\lambda)] = 0\end{aligned} \tag{71}$$

or:

$$\begin{aligned}\mathrm{Res}_\lambda\Bigg[\Bigg[\tau_{t_k}(\underline{t} - \underline{\epsilon}(\lambda))\,\tau_{t_k}(\underline{t}' + \underline{\epsilon}(\lambda)) + c\left(\lambda^{-1} + \frac{x' - x}{2}\right)(\tau_{t_k}(\underline{t} - \underline{\epsilon}(\lambda))\tau(\underline{t}' + \underline{\epsilon}(\lambda))\\ -\tau(\underline{t} - \underline{\epsilon}(\lambda))\tau_{t_k}(\underline{t}' + \underline{\epsilon}(\lambda))) + c^2(\lambda^{-1}(x - x') - \lambda^{-2})\\ \times\tau(\underline{t} - \underline{\epsilon}(\lambda))\tau(\underline{t}' + \underline{\epsilon}(\lambda))\Bigg]\exp\xi(\underline{t} - \underline{t}',\lambda)\Bigg] = 0\end{aligned} \tag{72}$$

which is an alternative bilinear form for the c-k constraint KP hierarchy (to be taken together with the bilinear form of the KP equation of course).

Expressing clearly this equation:

$$\mathrm{e}^{\sum_{i=1}^{\infty} y_i D_i} \sum_{j=0}^{\infty} p_j(-2\underline{y})[-\tfrac{1}{4} p_{j+1}(\tilde{D}) D_k^2 - c p_j(\tilde{D}) D_k - c y_1 p_{j+1}(\tilde{D}) D_k$$
$$-2c^2 y_1 p_j(\tilde{D}) - c^2 p_{j-1}(\tilde{D})]\tau\tau = 0 \qquad \forall \underline{y}. \tag{73}$$

The simplest non-trivial bilinear equation for τ contained in (73) is:

$$(4D_1 D_3 D_k^2 - 3D_2^2 D_k^2 - D_1^4 D_k^2 + 16c D_3 D_k - 16c D_1^3 D_k - 48c^2 D_1^2)\tau\tau = 0. \tag{74}$$

Taken together with the bilinear form of the KP equation (10), one finds for $k = 1$ in the field $v = \log\tau$:

$$v_{2t_2} - 4c v_{2x} - v_{4x} - 2v_{2x}^2 + \frac{v_{3x}^2 - v_{x,t_2}^2}{v_{2x} + 4c} = 0. \tag{75}$$

This equation was described in [19, 20] where it was called the non-local Boussinesq equation. Its soliton solutions were found to be '$pq = c$'-reductions of the KP N-soliton solutions [19]. In the next section we shall show how these results can be generalized to $p^k + c/p = q^k + c/q$ reductions.

A bilinear Bäcklund transformation and corresponding Lax pair can also be derived from relation (72).

4.3. Solutions

Theorem 3. The generalized k constrained KP hierarchy (defined by condition (63)) gives the following solutions ($q = \rho/\tau$ and $r = \sigma/\tau$):

$$\begin{aligned} \tau &= W(\varphi_1, \ldots, \varphi_N) \\ \rho &= \sqrt{c}\, W(\varphi_{1,x}, \ldots, \varphi_{N,x}) \\ \sigma &= \sqrt{c}\, W\left(\int^x \varphi_1, \ldots, \int^x \varphi_N\right) \end{aligned} \tag{76}$$

where $\varphi_i = a_i \exp\xi(\underline{t}, p_i) + b_i \exp\xi(\underline{t}, q_i)$ and with the relation $p_i^k + c/p_i = q_i^k + c/q_i$. $\int^x \varphi_i$ means $a_i/p_i \exp\xi(\underline{t}, p_i) + b_i/q_i \exp\xi(\underline{t}, q_i)$.

Proof. We will explicitly check the bilinear form (68) for these expressions. The proof is very similar to the one found in [10]. Since $\underline{\varphi}$ satisfies $\underline{\varphi}_{t_n} = \underline{\varphi}_{nx}$ one has:

$$\underline{\varphi}(\underline{t} - \underline{\epsilon}(\lambda)) = \underline{\varphi}(\underline{t}) - \lambda^{-1}\underline{\varphi}_x(\underline{t}) \qquad \underline{\varphi}(\underline{t} + \underline{\epsilon}(\lambda)) = \sum_{n\geqslant 0} \underline{\varphi}_{nx}(\underline{t})\lambda^{-n} \tag{77}$$

and hence (see [10] for details)

$$\begin{aligned} \tau(\underline{t} - \underline{\epsilon}(\lambda)) &= \sum_{j=0}^{N} (-\lambda)^{-j} |\underline{\varphi}, \ldots, \underline{\varphi}_{(N-j-1)x}, \underline{\varphi}_{(N-j+1)x}, \ldots, \underline{\varphi}_{Nx}| \\ \tau(\underline{t} + \underline{\epsilon}(\lambda)) &= \sum_{n=0}^{\infty} \lambda^{-n} |\underline{\varphi}, \underline{\varphi}_x, \ldots, \underline{\varphi}_{(N-2)x}, \underline{\varphi}_{(N-1+n)x}|. \end{aligned} \tag{78}$$

Substitution into the left-hand side of equation (68) yields (writing $\underline{\varphi}'$ for $\underline{\varphi}(\underline{t}')$):

$$\sum_{j=0}^{N} (-1)^j \sum_{n=0}^{\infty} p_n(\underline{t} - \underline{t}') |\underline{\varphi}', \underline{\varphi}'_x, \ldots, \underline{\varphi}'_{(N-2)x}, \underline{\varphi}'_{(N+k-j+n)x}|$$

$$
\begin{aligned}
&\times|\underline{\varphi},\ldots,\underline{\varphi}_{(N-j-1)x},\underline{\varphi}_{(N-j+1)x},\ldots,\underline{\varphi}_{Nx}|\\
&+c\sum_{j=0}^{N}(-1)^j\sum_{n=0}^{\infty}p_n(\underline{t}-\underline{t}')|\underline{\varphi}',\underline{\varphi}'_x,\ldots,\underline{\varphi}'_{(N-2)x},\underline{\varphi}'_{(N-j-1+n)x}|\\
&\times|\underline{\varphi},\ldots,\underline{\varphi}_{(N-j-1)x},\underline{\varphi}_{(N-j+1)x},\ldots,\underline{\varphi}_{Nx}|.
\end{aligned}
\tag{79}
$$

Since:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}p_n(\underline{t}-\underline{t}')\partial_x^n\underline{\varphi}'_{(N+k-j)x}=\underline{\varphi}_{(N+k-j)x}(\underline{t})\\
&\sum_{n=0,N-j-1+n\geqslant 0}^{\infty}p_n(\underline{t}-\underline{t}')\partial_x^n\underline{\varphi}'_{(N-j-1)x}=(\partial_x^{-1}\underline{\varphi})_{(N-j)x}-\delta_{Nj}(\partial_x^{-1}\underline{\varphi}')
\end{aligned}
\tag{80}
$$

expression (79) becomes:

$$
\begin{aligned}
&\sum_{j=0}^{N}(-1)^j|\underline{\varphi}',\underline{\varphi}'_x,\ldots,\underline{\varphi}'_{(N-2)x},(\underline{\varphi}_{(N+k-j)x}+c\underline{\varphi}_{(N-j-1)x}-c\delta_{jN}\partial_x^{-1}\underline{\varphi}')|\\
&\qquad\times|\underline{\varphi},\ldots,\underline{\varphi}_{(N-j-1)x},\underline{\varphi}_{(N-j+1)x},\ldots,\underline{\varphi}_{Nx}|
\end{aligned}
\tag{81}
$$

which equals:

$$
(-1)^{N+1}c|\underline{\varphi}_x\underline{\varphi}_{2x},\ldots,\underline{\varphi}_{Nx}||\underline{\varphi}',\underline{\varphi}'_x,\ldots,\underline{\varphi}'_{(N-2)x},\partial_x^{-1}\underline{\varphi}')|+|\underline{\varphi}',\underline{\varphi}'_x,\ldots,\underline{\varphi}'_{(N-2)x},\underline{\phi}|.
\tag{82}
$$

The first term in (82) is just $\rho(\underline{t})\sigma(\underline{t}')$, the second term contains the column vector $\underline{\phi}$ which is defined by:

$$
\phi_i\equiv\sum_{j=0}^{N}(-1)^j(\varphi_{i,(N+k-j)x}+c\partial_x^{-1}\varphi_{i,(N-j)x})|\underline{\varphi},\ldots,\underline{\varphi}_{(N-j-1)x},\underline{\varphi}_{(N-j+1)x},\ldots,\underline{\varphi}_{Nx}|.
\tag{83}
$$

Since $\partial_x^k\varphi_{i,(N-j)x}+c\partial_x^{-1}\varphi_{i,(N-j)x)}=(p_i^k+cp_i^{-1})\varphi_{i,(N-j)x}$, we find from (83) that $\phi_i=(p_i^k+cp_i^{-1})W(\varphi_1,\varphi_2,\ldots,\varphi_N,\varphi_i)=0$. This proves the theorem. □

A similar result can be obtained using a Darboux transformation [5, 16], starting from the trivial case $\tau=1$, $q=1$ and $r=c$. Notice that the condition $p_i^k+c/p_i=q_i^k+c/q_i$ becomes $(p_i-q_i)(p_iq_i-c)=0$ when $k=1$. Clearly this reduces to the $p_iq_i=c$ reduction [20, 21].

For determinants of type $\tau=W(\varphi_1,\ldots,\varphi_N)$, $\rho=\sqrt{c}W(\varphi_{1,x},\ldots,\varphi_{N,x})$, $\sigma=\sqrt{c}W(\int^x\varphi_1,\ldots,\int^x\varphi_N)$ (for any φ_i) one can compute $\hat\tau$:

$$
\hat\tau=cx\tau-c\tilde\tau+C\tau
\tag{84}
$$

where $\tilde\tau=\det[\int^x\underline{\varphi},\underline{\varphi}_x,\ldots,\underline{\varphi}_{(N-1)x}]$. In this case condition (70) becomes:

$$
\tau_{t_k}=-c\tilde\tau+C\tau.
\tag{85}
$$

5. Conclusions

We have examined the interpretation of the (m-vector) k constrained KP hierarchies as symmetry reductions on the KP hierarchy. It was found that in order to get unambiguous results, one needs to investigate their connection in terms of tau functions. The link between constraints and symmetry reductions was also found to be closely connected to a geometrical interpretation of these reductions.

In the last section, we proposed a generalization of the k constraint (only in the scalar case), allowing for reduced systems permitting N-soliton solutions with non-zero boundary conditions in the fields q and r (namely c an *a priori* introduced constant). An open question in this respect is the existence (in general) of rational solutions for these reductions and solutions to the m vector case ($m \geqslant 1$).

This generalized constraint (in the case $k = 1$) is closely linked to the so-called $pq = c$ reduction of the KP hierarchy. Hence the generalized 1-constraint is closely linked to the non-local Boussinesq hierarchy [19, 20]. For other values of k, we have a $p^k + c/p = q^k + c/q$ reduction.

This c-k constraint can also be interpreted as a symmetry constraint and has an accompanying geometrical interpretation for its tau function. From this property, an additional bilinear identity for the tau function of this hierarchy was easily derived.

Proving these results was achieved by using a squared eigenfunction potential method (recently introduced in [6]).

Acknowledgments

The authors are affiliated to the Fund for Scientific Research—Flanders (Belgium): RW as a postdoctoral fellow and IL as a research assistant. The authors wish to acknowledge financial support extended within the framework of the 'Interuniversitary Poles of Attraction Programme, Contract nr P4/08 - Belgian State, Prime Minister's Office - Federal Office for Scientific, Technical and Cultural Affairs'.

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